

Ricci curvature and the mechanics of solids

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Abstract

We discuss some differential geometry pertaining to continuum mechanics and the route recently taken by D.N. Arnold, R.S. Falk, and R. Winther in deriving new improved finite element schemes in linear elasticity from constructions in projective geometry.

Some vector analysis

We start with some basics from vector analysis [5]. Let us write

$$\partial_1 \equiv \partial/\partial x_1 \quad \partial_2 \equiv \partial/\partial x_2 \quad \partial_3 \equiv \partial/\partial x_3$$

for the partial derivatives in \mathbb{R}^3 . The gradient of a smooth function f defined on $U^{\text{open}} \subseteq \mathbb{R}^3$ is the vector field

$$\text{grad } f \equiv (\partial_1 f, \partial_2 f, \partial_3 f)$$

on U . If $X = (X_1, X_2, X_3)$ is a smooth vector field on U , then

$$\text{curl } X \equiv (\partial_2 X_3 - \partial_3 X_2, \partial_3 X_1 - \partial_1 X_3, \partial_1 X_2 - \partial_2 X_1).$$

It is readily verified that $\text{curl} \circ \text{grad} = 0$. Indeed, if U is a sufficiently simple set, such as a ball, then

$$\boxed{X = \text{grad } f, \text{ for some } f \iff \text{curl } X = 0.} \quad (1)$$

In effect, the curl of a vector field is the skew part of the 3×3 matrix $(\partial_i X_j)$ of partial derivatives. Let us instead consider the symmetric part

$$\Sigma = (\Sigma_{ij}) \equiv (\tfrac{1}{2}[\partial_i X_j + \partial_j X_i])$$

and ask for conditions that a given symmetric tensor field $\Sigma = (\Sigma_{ij})$ be of this form. The answer is that Σ should satisfy the *Saint-Venant* equations $\text{curl curl } \Sigma = 0$, where $\text{curl curl } \Sigma$ is the symmetric matrix obtained by

- firstly regarding Σ as a row vector (whose entries just happen to be column vectors) to form $\text{curl } \Sigma$,
- then regarding $\text{curl } \Sigma$ as a column vector (whose entries just happen to be row vectors) to form $\text{curl}(\text{curl } \Sigma)$.

The statement (1) has a useful counterpart as follows.

$$\boxed{\Sigma_{ij} = \tfrac{1}{2}[\partial_i X_j + \partial_j X_i], \text{ for some } X \iff \text{curl curl } \Sigma = 0.} \quad (2)$$

Indeed, we shall see that (2) can be deduced from (1). This, in turn, has consequences in the design of finite element schemes concerned with elasticity.

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Ricci curvature in three dimensions

Readers unfamiliar with differential geometry might omit this section on first reading. There is also a close link between (2) and Ricci curvature in three dimensions. Using the Einstein summation convention, if g_{ij} is a Riemannian metric with inverse g^{ij} , then the *Ricci* tensor R_{ij} is the symmetric tensor given by

$$\partial_k \Gamma_{ij}{}^k - \partial_i \Gamma_{jk}{}^k + \Gamma_{ij}{}^m \Gamma_{mk}{}^k - \Gamma_{ik}{}^m \Gamma_{jm}{}^k, \text{ where } \Gamma_{ij}{}^k \equiv \frac{1}{2} g^{kl} [\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}].$$

In three dimensions $R_{ij} = 0$ if and only if g_{ij} is flat, meaning that there is a local change of coördinates that transforms g_{ij} as a tensor into the flat metric δ_{ij} (more specifically, there is a change of coördinates with Jacobian matrix J such that $(g_{ij}) = J^t J$). The infinitesimal version of this statement is essentially (2). More precisely, if Σ_{ij} is an arbitrary symmetric tensor on $U^{\text{open}} \subseteq \mathbb{R}^3$ and we consider the metric $g_{ij}^\epsilon = \delta_{ij} + \epsilon \Sigma_{ij}$ where ϵ is sufficiently small that g_{ij}^ϵ is positive definite, then

$$(\text{curl curl } \Sigma)_{ij} = \frac{d}{d\epsilon} G_{ij}^\epsilon|_{\epsilon=0}$$

where G_{ij} is the Einstein tensor $Rg_{ij} - 2R_{ij}$ for $R = g^{kl} R_{kl}$. The Einstein tensor carries the same information as the Ricci tensor but has the advantage that the Bianchi identity simply says that G_{ij} is divergence-free $\nabla^i G_{ij} = 0$. Sure enough, one can readily verify that $\partial^i (\text{curl curl } \Sigma)_{ij} = 0$.

Translation from (1) to (2)

Firstly, some convenient notation in three dimensions. Let us write ϵ_{ijk} for the totally skew tensor with $\epsilon_{123} = 1$. It allows us to write $(\text{curl } X)_i = \epsilon_i{}^{jk} \partial_j X_k$. Now consider a pair $F = (X_\ell, Y_\ell)$ of vector fields on $U^{\text{open}} \subseteq \mathbb{R}^3$, regarded as a function with values in the vector space $\mathbb{W} \equiv \mathbb{R}^3 \oplus \mathbb{R}^3$. If we define the gradient of F by

$$\text{grad} \begin{bmatrix} X_\ell \\ Y_\ell \end{bmatrix} = \begin{bmatrix} \partial_j X_\ell - \epsilon_{j\ell}{}^m Y_m \\ \partial_j Y_\ell \end{bmatrix} \quad (3)$$

and use this definition naïvely to compute the curl of a vector field with values in \mathbb{W} , then we obtain

$$\text{curl} \begin{bmatrix} \Sigma_{j\ell} \\ \Xi_{j\ell} \end{bmatrix} = \begin{bmatrix} \epsilon_i{}^{jk} \partial_j \Sigma_{k\ell} - \epsilon_i{}^{jk} \epsilon_{j\ell}{}^m \Xi_{km} \\ \epsilon_i{}^{jk} \partial_j \Xi_{k\ell} \end{bmatrix} = \begin{bmatrix} \epsilon_i{}^{jk} \partial_j \Sigma_{k\ell} - \Xi_{\ell i} + \delta_{i\ell} \Xi_m{}^m \\ \epsilon_i{}^{jk} \partial_j \Xi_{k\ell} \end{bmatrix}.$$

It is readily verified that $\text{curl} \circ \text{grad} = 0$. This says precisely that (3) defines a flat connection, which enables one to deduce that if U is a sufficiently simple set, such as a ball, then

$$\boxed{\Psi = \text{grad } F, \text{ for some } F \iff \text{curl } \Psi = 0.} \quad (4)$$

To deduce (2), let us suppose that Σ_{ij} is symmetric and set

$$\Psi = \begin{bmatrix} \Sigma_{j\ell} \\ \Xi_{j\ell} \end{bmatrix} = \begin{bmatrix} \Sigma_{j\ell} \\ \epsilon_{\ell}^{im} \partial_i \Sigma_{mj} \end{bmatrix}, \text{ so that } \text{curl } \Psi = \begin{bmatrix} 0 \\ (\text{curl curl } \Sigma)_{i\ell} \end{bmatrix}.$$

If $\text{curl curl } \Sigma = 0$, we immediately infer the existence of vector fields X_ℓ and Y_ℓ on U such that

$$\text{grad} \begin{bmatrix} X_\ell \\ Y_\ell \end{bmatrix} = \Psi \quad \text{i.e.} \quad \begin{bmatrix} \partial_j X_\ell - \epsilon_{j\ell}^m Y_m \\ \partial_j Y_\ell \end{bmatrix} = \begin{bmatrix} \Sigma_{j\ell} \\ \epsilon_{\ell}^{im} \partial_i \Sigma_{mj} \end{bmatrix}.$$

In particular, $\Sigma_{j\ell} = \frac{1}{2}[\partial_j X_\ell + \partial_\ell X_j]$, as required.

Continuum mechanics

Although different words are used, Riemannian differential geometry in three dimensions is exactly what is needed to set up the mechanics of solids [4]. The metric tensor is known as the *strain* in continuum mechanics. The Einstein tensor is known as the *stress*. The Bianchi identity says that the stress tensor is divergence-free, interpreted as a *conservation law* in mechanics. Linearising around the flat metric gives the following complex of tensors on \mathbb{R}^3

$$\begin{array}{ccccccc} X_i & \mapsto & \frac{1}{2}[\partial_j X_j + \partial_j X_i] & & S_{ij} & \mapsto & \partial^i S_{ij} \\ \text{displacement} & \rightarrow & \text{strain} & \rightarrow & \text{stress} & \rightarrow & \text{load}, \quad (5) \\ & & \Sigma_{ij} & \mapsto & \epsilon_i^{km} \epsilon_j^{\ell n} \partial_k \partial_\ell \Sigma_{mn} & & \end{array}$$

where the *displacement* and *load* are vector fields whilst the stress and strain are symmetric 2-tensors.

Finite element schemes

We have already seen that the flat connection (3) somehow embodies the operator $\Sigma \mapsto \text{curl curl } \Sigma$ relating strain and stress in (5). More generally and precisely, the whole complex (5) may be derived from the connection (3). To do this, recall that the gradient operator (3) concerned functions with values in $\mathbb{W} = \mathbb{R}^3 \oplus \mathbb{R}^3$. Thus, we may write

$$\begin{array}{ccccccc} \mathbb{W} & \xrightarrow{\text{grad}} & \mathbb{R}^3 \otimes \mathbb{W} & \xrightarrow{\text{curl}} & \mathbb{R}^3 \otimes \mathbb{W} & \xrightarrow{\text{div}} & \mathbb{W} \\ \parallel & & \parallel & & \parallel & & \parallel \\ \mathbb{R}^3 & & \boxed{\mathbb{R}^3} \oplus \mathbb{S}^2 \mathbb{R}^3 & & \boxed{\mathbb{R}^3 \otimes \mathbb{R}^3} & & \boxed{\mathbb{R}^3} \\ \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus \\ \boxed{\mathbb{R}^3} & & \boxed{\mathbb{R}^3 \otimes \mathbb{R}^3} & & \mathbb{S}^2 \mathbb{R}^3 \oplus \boxed{\mathbb{R}^3} & & \mathbb{R}^3 \end{array} \quad (6)$$

where $\mathbb{S}^2 \mathbb{R}^2$ denotes symmetric 3-tensors whilst skew 3-tensors are identified with \mathbb{R}^3 using ϵ_i^{jk} . In this diagram, the spaces indicated thus $\boxed{\phantom{\mathbb{R}^3}}$ are joined by isomorphisms indicated thus \nearrow . A simple diagram chase cancels these

spaces and results in the linear elasticity complex (5). In [1], Arnold, Falk, and Winther use a halfway-house complex

$$\begin{array}{ccccccc} \mathbb{R}^3 & & & & & & \mathbb{R}^3 \\ \oplus & \longrightarrow & \mathbb{R}^3 \oplus \mathbb{S}^2\mathbb{R}^3 & \longrightarrow & \mathbb{S}^2\mathbb{R}^3 \oplus \mathbb{R}^3 & \longrightarrow & \oplus \\ \mathbb{R}^3 & & & & & & \mathbb{R}^3 \end{array}$$

obtained by cancelling only $\boxed{\mathbb{R}^3 \otimes \mathbb{R}^3} \nearrow \boxed{\mathbb{R}^3 \otimes \mathbb{R}^3}$ from (6), to construct new and stable finite element schemes for linear elasticity mimicking the previously known stable finite element schemes for the grad-curl-div complex.

Projective geometry

The connection (3) may be viewed as follows. Consider the unit three-sphere $S^3 \subset \mathbb{R}^4$. There is no difficulty in taking the gradient of a function F on S^3 with values in the skew 2-tensors $\Lambda^2\mathbb{R}^4$. However, each point on S^3 is also a vector $v \in \mathbb{R}^4$, which may be used to decompose these skew 2-tensors:–

$$\Lambda^2\mathbb{R}^4 = \{\omega \text{ s.t. } v \wedge \omega = 0\} \oplus \{\omega \text{ s.t. } v \lrcorner \omega = 0\} \cong \mathbb{R}^3 \oplus \mathbb{R}^3.$$

This decomposition does not see the sign of v and so descends to the quotient of S^3 under antipodal identification, namely real projective 3-space \mathbb{RP}_3 . The upshot is that the gradient of F may be written in terms of the intrinsic calculus on \mathbb{RP}_3 and viewed in a standard affine coördinate patch $\mathbb{R}^3 \hookrightarrow \mathbb{RP}_3$. The result is (3). The construction of (5) from (6) is due to Calabi [2]. It may also be viewed as a geometric realisation of the Jantzen-Zuckerman translation principle from representation theory [6] and, as such, admits vast generalisation in the newly developed field of parabolic geometry [3].

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